

# Whirly 3-Interval Exchange Transformations

YUE WU<sup>1</sup>

3519 HEARTLAND KEY LN, KATY, TX, 77494

WUUYUE@GMAIL.COM

## Abstract

Irreducible interval exchange transformations are studied with regarding to whirly, a condition for non-trivial spatial factor. Uniformly whirly transform is defined and claimed to be residue in the automorphism space. An equivalent condition is introduced for whirly transformation. We will prove that almost all 3-interval exchange transformations are whirly, which is a constructional approach with deep investigation to the Rauzy-Veech Induction.

## 1 Introduction

An interval exchange transformation perturbs the half-closed half-open subintervals of a half-closed half-open interval. The subintervals have lengths corresponding to the vector  $\lambda = (\lambda_1, \dots, \lambda_m)$ ,  $\lambda_i > 0$ ,  $1 \leq i \leq m$ .

All such vectors form a positive cone  $\Lambda_m \subset R^m$ . The subintervals thus are  $[\beta_{i-1}, \beta_i)$ ,  $1 \leq i \leq m$ , with  $\bigcup [\beta_{i-1}, \beta_i) = [0, |\lambda|)$ , where

$$(1.1) \quad \begin{aligned} |\lambda| &= \sum_{i=1}^m \lambda_i \\ \text{and,} \\ \beta_i(\lambda) &= \begin{cases} 0 & i = 0 \\ \sum_{t=1}^i \lambda_t & 1 \leq i \leq m \end{cases} \end{aligned}$$

Let  $\mathcal{G}_m$  be the group of  $m$ -permutation, and  $\mathcal{G}_m^0$  be the subset of  $\mathcal{G}_m$  which contains all the irreducible permutations on  $\{1, 2, \dots, m\}$ . A permutation  $\pi$  is irreducible if and only if for any  $1 \leq k < m$ ,  $\{1, 2, \dots, k\} \neq \{\pi(1), \dots, \pi(k)\}$ , or equivalently  $\sum_{i=1}^k (\pi(i) - i) > 0$ , ( $1 \leq k < m$ ). Given  $\lambda \in \Lambda_m$ ,  $\pi \in \mathcal{G}_m^0$ , the corresponding interval exchange transformation is defined by:

$$(1.2) \quad \begin{aligned} T_{\lambda, \pi} x &= x - \beta_{i-1}(x) + \beta_{\pi(i-1)}(\lambda^\pi), \quad (x \in [\beta_{i-1}(\lambda), \beta_i(\lambda))) \\ \text{where } \lambda^\pi &= (\lambda_{\pi^{-1}1}, \lambda_{\pi^{-1}2}, \dots, \lambda_{\pi^{-1}m}). \end{aligned}$$

Obviously  $\beta_{\pi(i-1)}(\lambda^\pi) = \sum_{t=1}^{\pi(i-1)} \lambda_{\pi^{-1}t}$ , and the transformation  $T_{\lambda, \pi}$ , which is also denoted by  $(\lambda, \pi)$ , sends the  $i$ th interval to the  $\pi(i)$ th position.

In M.Keane[11], the i.d.o.c.(infinite distinct orbits condition) is raised for the equivalent condition of minimality.  $\lambda, \pi$  is said to satisfy the *i.d.o.c.* if

<sup>1</sup>Schlumberger PTS Full Waveform Inversion Center of Excellence, Houston, Texas, USA

i) for any  $0 \leq i < m$ ,  $\{T^k \beta_i, k \in \mathbb{Z}\}$  is a infinite set;

ii)  $\{T^k \beta_i, k \in \mathbb{Z}\} \cap \{T^k \beta_j, k \in \mathbb{Z}\} = \Phi$ , whenever  $i \neq j$ .

Suppose  $m > 1$ ,  $(\lambda, \pi) \in \Lambda_m \times \mathcal{G}_m^*$ , where  $\mathcal{G}_m^*$  is the set of irreducible permutations with the property that  $\pi(j+1) \neq \pi(j)+1$  for all  $1 \leq j \leq m-1$ . Let  $I$  be an interval of the form  $I = [\xi, \eta)$ ,  $0 \leq \xi < \eta \leq |\lambda|$ . Since  $T$  is defined on  $[0, \lambda)$ , and  $T$  is Lebesgue measure preserving, we know that Lebesgue almost all points of  $I$  return to  $I$  infinitely often under iteration of  $T$ . We use  $T|_I$  to denote the induced transformation of  $T$  on  $I$ . By W.A. Veech[17],  $T|_I$  is an interval exchange transformation with  $(m-2)$ ,  $(m-1)$ , or  $m$  discontinuities.

**Definition 1.1** (Admissible Interval). *Suppose  $(\lambda, \pi)$  satisfies the i.d.o.c., and  $I = (\xi, \eta)$  where  $\xi = T^k \beta_s, (1 \leq s < m)$ ;  $\eta = T^l \beta_t, (1 \leq t < m)$  and  $\tau \in \{k, l\}$  have the following property: If  $\tau \geq 0$ , there is no  $j$ ,  $0 < j < \tau$ , such that  $T^j \beta_s \in I$ ; If  $\tau < 0$ , there is no  $j$ ,  $0 \geq j > \tau$ , such that  $T^j \beta_s \in I$ . Then we say that  $I$  is an admissible subinterval of  $(\lambda, \pi)$ . [VEE4].*

$[0, \beta_i)$ ,  $(0 < i < m)$  and  $[0, T\beta_i)$ ,  $(1 \leq i < m, \pi(i+1) \neq 1)$  are admissible.

**Rauzy-Veech induction.** For  $(\lambda, \pi)$ , the Rauzy map sends it to the induced map on  $[0, |\lambda| - \min\{\lambda_m, \lambda_{\pi^{-1}m}\})$ , which is the largest admissible interval of form  $J = [0, L)$ ,  $0 < L < |\lambda|$ .

Given any permutation, two action  $a$  and  $b$  are:

$$(1.3) \quad a(\pi)(i) = \begin{cases} \pi(i) & i \leq \pi^{-1}m \\ \pi(i-1) & \pi^{-1}m+1 < i \leq m \\ \pi(m) & i = \pi^{-1}m+1 \end{cases}$$

and

$$(1.4) \quad b(\pi)(i) = \begin{cases} \pi(i) & \pi(i) \leq \pi(m) \\ \pi(i)+1 & \pi(m)+1 < \pi(i) < m \\ \pi(m)+1 & \pi(i) = m \end{cases}$$

The Rauzy-Veech map  $\mathcal{Z}(\lambda, \pi) : \Lambda_m \times \mathcal{G}_m^0 \rightarrow \Lambda_m \times \mathcal{G}_m^0$  is determined by :

$$(1.5) \quad \mathcal{Z}(\lambda, \pi) = (A(\pi, c)^{-1} \lambda, c\pi)$$

where  $c = c(\lambda, \pi)$  is defined by

$$(1.6) \quad c(\lambda, \pi) = \begin{cases} a, & \lambda_m < \lambda_{\pi^{-1}m} \\ b, & \lambda_m > \lambda_{\pi^{-1}m}. \end{cases}$$

$\mathcal{Z}(\lambda, \pi)$  is a.e. defined on  $\Lambda_m \times \{\pi\}$ , for each  $\pi \in \mathcal{G}_m^0$ .

The matrices  $A = A(\pi, c)$  in 1.5 are defined as the following:

$$(1.7) \quad A(\pi, a) = \left( \begin{array}{c|ccccc} & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & 0 & 0 \\ & \cdot & \cdot & \cdots & \cdot & \cdot \\ I_{\pi^{-1}m} & 0 & 0 & \cdots & 0 & 0 \\ & 1 & 0 & \cdots & 0 & 0 \\ \hline & 0 & 1 & \cdots & 0 & 1 \\ & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdot & \cdot & \cdots & \cdot & \cdot \\ & 0 & 0 & \cdots & 0 & 1 \\ & 1 & 0 & \cdots & 1 & 0 \end{array} \right)$$

$$(1.8) \quad A(\pi, b) = \left( \begin{array}{c|c} I_{m-1} & 0 \\ \hline 0 & 1 \end{array} \right)$$

$\underbrace{\quad\quad\quad}_{1 \text{ at the } j\text{th position}}$

where  $I_k$  is the  $k$ -identity matrix, and  $j = \pi^{-1}m$ .

And the normalized Rauzy map  $\mathcal{R} : \Delta_{m-1} \times \mathcal{G}_m^0 \rightarrow \Delta_{m-1} \times \mathcal{G}_m^0$  is defined by

$$(1.9) \quad \mathcal{R}(\lambda\pi) = \left( \frac{A(\pi, c)^{-1}\lambda}{|A(\pi, c)^{-1}\lambda|}, c\pi \right) = \left( \frac{\pi_1^* \mathcal{Z}(\lambda, \pi)}{|\pi_1^* \mathcal{Z}(\lambda, \pi)|}, \pi_2^* \mathcal{Z}(\lambda, \pi) \right).$$

where  $\pi_1^*$  and  $\pi_2^*$  are the projection to the first coordinate and the second coordinate resp.. Iteratively,

$$(1.10) \quad \mathcal{Z}^n(\lambda, \pi) = ((A^{(n)})^{-1}\lambda, c^{(n)}\pi) = (\lambda^{(n)}, \pi^{(n)})$$

here

$$(1.11) \quad c^{(n)} = c_n c_{n-1} \cdots c_1, (c_1, \dots, c_n \in \{a, b\}, c_i = c(\mathcal{Z}^{i-1}(\lambda, \pi)))$$

and

$$(1.12) \quad A^{(n)} = A(\pi, c_1)A(c^{(1)}\pi, c_2)A(c^{(2)}\pi, c_3) \cdots A(c^{(n-1)}\pi, c_n)$$

The Rauzy class  $\mathcal{C} \subseteq \mathcal{G}_m$  of  $\pi$  is a set of orbits for the group of maps generated by  $a$  and  $b$ . On the  $\mathcal{R}$  invariant component  $\Delta_{m-1} \times \mathcal{C}$ , we have:

**Theorem 1.2** (H.Masur[13];W.A. Veech[15]). *Let  $\pi \in \mathcal{G}_m^0$ , the irreducible permutations. For Lebesgue almost all  $\lambda \in \Lambda_m$  normalized Lebesgue measure on  $I^\lambda$  is the unique invariant Borel probability measure for  $T_{(\lambda, \pi)}$ . In particular,  $T_{(\lambda, \pi)}$  is ergodic for almost all  $\lambda$ .*

**Whirly Action, Whirly Transformation.** In this paper, weak topology is assumed (coincides with  $L_p$  topology since the space is bounded).

In [7], whirly action is introduced by E. Glasner, B. Tsirelson, B. Weiss. The purpose is to study the condition for a Polish group action to admit spatial model[Glasner, Weiss]. In the same paper, they showed that in the group  $G$  of automorphisms on a finite Lebesgue space, whirly (in the sense of  $Z$  action) is a topological generic property, i.e. the set of whirly automorphism is residual in  $G$ . The concept of 'whirly transformation' is inherited from the theory about general group action, and implies weak mixing. It is interesting to ask whether whirly is a generic property in the space of interval exchange transformations. For which we give a positive answer for three interval exchange transformations as Theorem 1.15 in this paper.

**Definition 1.3.** [Near Action][E. Glasner, B. Tsirelson, B. Weiss[7]] Suppose  $\mathbb{P}$  is a Polish group and  $(\mathbb{X}, \mathcal{B}, \mu)$  is a standard probability Borel space. We say a Borel map  $\mathbb{P} \times \mathbb{X} \rightarrow \mathbb{X}$   $((g, x) \rightarrow gx)$  is a near action of  $\mathbb{P}$  on  $(\mathbb{X}, \mathcal{B}, \mu)$  if it satisfies the following properties:

- (i)  $ex = x$  for a.e.  $x \in \mathbb{X}$ , where  $e$  is the identity element of  $\mathbb{P}$ ;
- (ii)  $g(hx) = (ghx)$  for a.e.  $x \in \mathbb{X}$ , where  $g, h \in \mathbb{P}$ ;
- (iii) Each  $g \in G$  preserves the measure  $\mu$ .

**Note.** the set of measure one in Definition 1.3 (ii) may depend on the pair  $g, h$ . It is easy to see that a near action is continuous homomorphism from  $\mathbb{P}$  to  $G$  ( $G$  is the automorphism group of  $\mathbb{X}$ , see Chapter1 Section1).

Without considering the measure, we have the Borel action, satisfying similar condition as in definition 4.0.1, defined as below:

**Definition 1.4.** [Borel Action] Suppose  $\mathbb{P}$  is a Polish group and  $(\mathbb{X}, \mathcal{B}, \mu)$  is a standard probability Borel space, we say a Borel map  $\mathbb{P} \times \mathbb{X} \rightarrow \mathbb{X} ((g, x) \rightarrow gx)$  is a near action of  $\mathbb{P}$  on  $(\mathbb{X}, \mathcal{B}, \mu)$  if it satisfies the following properties:

- (i)  $ex = x$  for all  $x \in \mathbb{X}$ , where  $e$  is the identity element of  $\mathbb{P}$ ;
- (ii)  $g(hx) = (ghx)$  for all  $x \in \mathbb{X}$ , where  $g, h \in \mathbb{P}$ .

It is interesting to ask: given a near action, can we lift the mod 0 action to a Borel action, or just ask can we define  $g(x)$  and keep the homomorphic property everywhere.

**Definition 1.5.** [Spatial  $\mathbb{P}$  action][E. Glasner, B. Tsirelson, B. Weiss[7]] A spatial  $G$ -action on a standard Lebesgue space  $(\mathbb{X}, \mathcal{B}, \mu)$  is a Borel action of  $\mathbb{P}$  on the space such that each  $g \in \mathbb{P}$  preserves  $\mu$ . In the sense that a spatial action is itself a near action, it is the spatial model of the near action.

The opposite end of spatial  $\mathbb{P}$  action is 'whirly action' defined by:

**Definition 1.6.** [Whirly Action][E. Glasner, B. Tsirelson, B. Weiss[7]] Given  $\varepsilon > 0$ , if for all sets  $A, B \in \mathcal{B}$  with  $\mu(A), \mu(B) > 0$ , there exists  $\gamma \in N_\varepsilon(Id)$  (the  $\varepsilon$  neighborhood of the identity  $Id = e$  in  $\mathbb{P}$ ), such that  $\mu(A \cap gB)$  then we say the near action of  $\mathbb{P}$  on  $(\mathbb{X}, \mathcal{B}, \mu)$  is *whirly*.

**Theorem 1.7.** [E. Glasner, B. Tsirelson, B. Weiss[7]] A *whirly* action cant have nontrivial spatial factor, thus has no spatial model.

**Remark 1.8.** If an automorphism  $(\mathbb{X}, \mathcal{B}, T, \mu)$  is rigid, then its weak closure  $Wcl(T)$  is a closed subgroup of  $G = Aut(\mathbb{X}, \mathcal{B}, \mu)$ . With the induced topology,  $Wcl(T)$  is also a Polish space. Based on this fact, the *whirly* transformation is a concept induced from *whirly* action.

Let  $(\mathbb{X}, \mathcal{B}, \mu)$  be the standard Lebesgue probability space,  $\mathbb{X} = [0, 1]$ , and denote  $G = Aut(\mathbb{X})$  the Polish group of its automorphism.

**Definition 1.9** (Whirly Automorphism). We say a rigid system  $(X, \mathcal{B}, \mu, T)$  is *whirly*, if given  $\epsilon > 0$  for any  $\mu$  positive measure sets  $A$  and  $B$  ( $\mu(A), \mu(B) > 0$ ) in  $\mathcal{B}$ , there exists  $n$  such that  $T^n \in U_\epsilon$  (the  $\epsilon$ -neighborhood of identity map in the weak topology of  $G$ ), and  $\mu(T^n A \cap B) > 0$ .

Whirly implies rigid. It is showed in E.Glasner, B.Weiss[8] that if  $(\mathbb{X}, \mathcal{B}, \mu, T)$  is *whirly* then it is weak mixing. In the same paper, it is proved that:

**Theorem 1.10.** [E.Glasner, B.Weiss[8]] The set of all the *whirly* transformations is residual in  $G$ .

Next we define uniformly *whirly*, which is stronger than *whirly*:

**Definition 1.11** (Uniformly Whirly). A rigid system  $(\mathbb{X}, \mathcal{B}, \mu, T)$  is *uniformly whirly* if given  $\varepsilon > 0$  for any  $0 < \alpha, \beta < 1$ , we have

$$\inf_{\mu(A)=\alpha, \mu(B)=\beta} \sup_{T^n \in U_\varepsilon} \{\mu(T^n A \cap B)\} > 0$$

**Theorem 1.12.** The set of uniformly *whirly* automorphisms is residue in  $G$ .

Next we introduce another way to define the concept of *whirly* automorphism and verify the equivalence:

**Definition 1.13** (Whirly Automorphism). *A rigid ergodic automorphism  $T \in G$  is said to be whirly if given  $\varepsilon > 0$ , for any  $l \in \mathbb{N}$  (or  $-l \in \mathbb{N}$ ) and a  $\mu$ -positive measure set  $A \in \mathcal{B}$ , there exists  $n \in \mathbb{N}$  such that  $T^n \in U_\varepsilon$ , and  $\mu(T^n A \cap T^l A) > 0$ .*

**Theorem 1.14.** *Conditions in Definition 1.9 and Definition 1.13 for a transformation to be whirly are equivalent to each other.*

*Proof.* Suppose  $T \in G$  satisfies the condition in Definition 1.13 (w.l.o.g., we take the case that  $-l \in \mathbb{N}$ ), then we claim that for any  $A, B \in \mathcal{B}$  with  $\mu(A), \mu(B) > 0$  we have there exists  $n \in \mathbb{N}$  such that  $T^n \in U_\varepsilon$  and  $\mu(T^n A \cap B) > 0$ . Since  $T$  is ergodic, there exist  $-q \in \mathbb{N}$  such that

$$\mu(T^q A \cap B) > 0$$

Then

$$\mu(A \cap T^{-q} B) > 0$$

therefore there exists  $n \in \mathbb{N}$  such that  $T^n \in U_\varepsilon$  and

$$\mu(T^n(A \cap T^{-q} B) \cap T^m(A \cap T^{-q} B)) > 0$$

Thus  $\mu(T^n A \cap B) > 0$

The opsite direction is obvious. □

We will prove the following theorem in section 3, which is a part of the author's 2006 Ph.D. thesis[18]:

**Theorem 1.15.** *For Lebesgue almost all  $\lambda \in \Lambda_3$ ,  $T_{(\lambda, \pi)}$  is whirly.*

## 2 Three Interval Exchange Transformation Space

In W.A.Veech[15], the theory about interval exchange transformation space is established. We will utilize the result in [VEE1]. Let  $m > 1$ , and  $\pi$  be the symmetric permutation (i.e.  $\pi = (m, m-1, \dots, 1)$ ). In W.A.Veech[15] it is proved that for almost every  $\lambda$  the induced transformation of  $T_{\lambda, \pi}$  on  $[0, \max\{\lambda_1, \lambda_m\})$ . is an  $(\alpha, \pi)$  interval exchange transformation with  $|\alpha| = \max\{\lambda_1, \lambda_m\}$  and  $\pi$  still the same permutation. That is a transformation  $\mathcal{T}_2 : \lambda \rightarrow \frac{\alpha}{|\alpha|}$ , since the permutations are the same, we may understand  $\mathcal{T}_2(\lambda) \sim \lambda \times \{\pi\} \rightarrow \frac{\alpha}{|\alpha|} \times \{\pi\}$ . So without confusing, let  $\mathcal{T}_2(\lambda, \pi) = \mathcal{T}_2(\lambda) \times \{\pi\}$ . When  $m = 3$ ,  $f_2(\lambda) = (\frac{1}{1-\lambda_1} + \frac{1}{1-\lambda_3}) \prod_{j=1}^2 \frac{1}{\lambda_j + \lambda_{j+1}}$  is the density of a conservative ergodic invariant measure for  $\mathcal{T}_2$  by [VEECH].

We claim that if  $(\lambda, \pi)$  satisfies i.d.o.c. ,  $\pi(j) = m - j + 1$ ,  $\mathcal{T}_2(\lambda, \pi)$  , for some  $k$  such that  $\mathcal{Z}^k(\lambda, \pi) = (\alpha, \pi)$  with  $|\alpha| = \max\{\lambda_1, \lambda_m\}$ . To verify this we need the following lemma:

**Lemma 2.1.**  $\lambda \in \Lambda_{m-1}, m \geq 3, T_{(\lambda, \pi)}$  satisfies i.d.o.c. , and  $\mathcal{Z}^k(\lambda, \pi) = (\lambda', \pi')$ ,  $|\lambda'| > \max\{\lambda_1, \lambda_m\}$ , then  $[0, \max\{\lambda_1, \lambda_m\})$  is an admissible interval of  $(\lambda', \pi')$ .

*Proof.*  $T_{(\lambda', \pi')}$  is the induced transformation of  $T_{\lambda, \pi}$  on  $[0, |\lambda'|)$ . If  $\lambda_1, \lambda_m$  then  $\lambda_1$  is a discontinuous point of  $T_{(\lambda', \pi')}$ ,  $[0, \lambda_1)$  is an admissible interval of  $(\lambda', \pi')$ , if  $\lambda_m > \lambda_1$ , then since  $\lambda_m = T_{(\lambda, \pi)}(\beta_{m-2}(\lambda))$ ,  $\lambda_m$  is in the interior of a subinterval determined by  $\lambda$ , i.e.  $\lambda_m \in [\beta_i, \beta_{i+1})$ , for some  $1 \leq i < m$ . Therefore  $\beta_{m-1} \in T[\beta_i, \beta_{i+1})$ ,  $\lambda_m = T(\beta_{m-1}) \in T^2[\beta_i, \beta_{i+1})$ , that means  $\lambda_m$  is a discontinuous point of  $T_{\lambda', \pi'}$ . By the definition of admissible interval,  $[0, \lambda_m)$  is an admissible interval associated with  $T_{\lambda', \pi'}$ . □

**Proposition 2.2.** Suppose  $\lambda \in \Lambda_{m-1}$ ,  $\pi(j) = m - j + 1$ ,  $1 \leq j \leq m$ , and  $(\lambda, \pi)$  satisfies i.d.o.c. . Then there exists  $k_0 \in \mathbb{N}$  such that  $\mathcal{Z}^{k_0}(\lambda, \pi) = (\alpha, \pi)$ , where  $|\alpha| = \max\{\lambda_1, \lambda_m\}$ . Therefore,  $\mathcal{T}_2(\lambda, \pi) = \mathcal{R}^{k_0}(\lambda, \pi)$ .

*Proof.* Assume for all  $k \in \mathbb{N}$ ,  $\mathcal{Z}^k(\lambda, \pi) = (\alpha^k, \pi)$  such that  $|\alpha|^k \neq \max\{\lambda_1, \lambda_m\}$ . Since  $\lambda, \pi$  satisfies i.d.o.c. ,  $|\pi_1^*(\mathcal{Z}^k(\lambda, \pi))| \rightarrow 0$  as  $k \rightarrow \infty$ , there exist  $k \geq 0$  such that  $|\pi_1^*(\mathcal{Z}^{k_0}(\lambda, \pi))| > \max\{\lambda_1, \lambda_m\}$ , and  $|\pi_1^*(\mathcal{Z}^{k_0+1}(\lambda, \pi))| < \max\{\lambda_1, \lambda_m\}$ . By Lemma 4.3.1 for any  $r > |\pi_1^*(\mathcal{Z}^{k_0+1}(\lambda, \pi))|$ ,  $[0, r)$  is not an admissible interval of  $(\mathcal{Z}^{k_0}(\lambda, \pi))$ , that is a contradiction to the fact that  $[0, \max\{\lambda_1, \lambda_m\})$  is an admissible interval of  $(\lambda', \pi')$ .  $\square$

The above argument assures us that some general result about the iteration of Rauzy-Veech induction may be applied to  $\mathcal{T}_2$ . For convenience, let's denote the  $T_{\lambda, \pi}$  induced map on  $[0, \max\{\lambda_1, \lambda_m\})$  by  $(\alpha, \pi)$ , define  $\mathcal{Z}_* : \Lambda_m \times \{\pi\} \rightarrow \Lambda_m \times \{\pi\}$  by  $\mathcal{Z}_*(\lambda, \pi) = (\alpha, \pi)$  with  $|\alpha| = \max\{\lambda_1, \lambda_m\}$ .

Next we limit the discussion to the case  $m = 3$ . Recall Section 1 for the visitation matrix associated with  $\mathcal{Z}^n(\lambda, \pi)$ . That is to say if  $\mathcal{Z}^n(\lambda, \pi) = (\alpha^{(n)}, \pi)$ , then  $\lambda = A^{(n)}$ ,  $a_i^{(n)}$  is the first return time of the  $i$ -th subinterval of  $[0, |\alpha^{(n)}|)$  under  $T_{(\lambda, \pi)}$ . It will be shown that for all  $n \in \mathbb{N}$ ,  $a_2^{(n)} = a_1^{(n)} + a_3^{(n)} - 1$ . In fact we will verify the same equality for a more general case. It is done by looking at the Rauzy graph for the closed paths based at  $\pi = (3, 2, 1)$ . The Rauzy class of  $\pi = (3, 2, 1)$  is  $\{\pi, \pi_1, \pi_2 | \pi_1 = a\pi = (3, 1, 2), \pi_2 = b\pi = (2, 3, 1)\}$ . The Rauzy graph of  $\pi$  is the following:

$$\begin{aligned} A(\pi, a) = A(\pi_1, a) &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ A(\pi, b) = A(\pi_1, b) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ A(\pi_2, a) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ A(\pi_2, b) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{aligned}$$

**Lemma 2.3.** If  $\mathcal{Z}_2(\lambda, \pi) = (\alpha, \pi)$ ,  $\lambda \in \Lambda_3$ ,  $\pi = (3, 2, 1)$ , and the visitation matrix is  $A$  ( $\lambda = A\alpha$ ). Then  $a_2 + 1 = a_1 + a_3$ .

*Proof.* The proof is splitted into two cases:

**Case 1.**  $[ab^l a \text{ or } ba^l b]$

1. Star from  $\pi$ , go along the path  $ab^l a$ , come back to  $\pi$ , the associated visitation matrix is  $A^{(l+2)}$ , we want to show that :

$$a_2^{(l+2)} = a_1^{(l+2)} + a_3^{(l+2)} - 1$$

Since

$$A^{(1)} = A(\pi, a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$a_1^{(1)} = a_3^{(1)} = 1, a_2^{(1)} = 2$$

$$\begin{aligned} A^{(2)} &= A^{(1)} \cdot A(\pi, b) = (A_1^{(1)} A_2^{(1)} A_3^{(1)}) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= (A_1^{(1)} + A_3^{(1)}, A_2^{(1)}, A_3^{(1)}) \end{aligned}$$

(where  $A_i^{(n)}$  is the  $i$ -th column vector of  $A^{(n)}$ ).

.....

$$A^{(l+1)} = (A_1^{(1)} + lA_3^{(1)}, A_2^{(1)} A_3^{(1)})$$

$$\begin{aligned} (2.1) \quad A^{(l+2)} &= A^{(l+1)} \cdot A(\pi_1, a) \\ &= (A^{(1)} 1 + lA_3^{(1)}, A_2^{(1)}, A_3^{(1)}) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= (A_1^{(1)} + lA_3^{(1)}, A_1^{(1)} + (l+1)A_3^{(1)}, A_2^{(1)}) \end{aligned}$$

Therefore

$$\begin{aligned} (2.2) \quad a_1^{(l+2)} &= a_1^{(1)} + la_3^{(1)} = l+1 \\ a_2^{(l+2)} &= a_1^{(1)} + (l+1)a_3^{(1)} = l+2 \\ a_3^{(l+2)} &= a_2^{(1)} = 2 \end{aligned}$$

Thus  $a_2 = a_1 + a_3 - 1$  is proved for path  $ab^l a$ .

2. Similarly as 1. we get the following:

if instead of going along  $ab^l a$  as in (i), but going along the closed path  $ba^l b$  the associated matrix is  $A^{(k+2)}$ , then we have

$$a_2^{(l+2)} = a_1^{(l+2)} + a_3^{(l+2)} - 1$$

**Case 2.**  $[p_0 ab^l a$  or  $p_0 ba^l b]$

1. Suppose the closed path is  $p = p_0 ab^l a$ , where  $p_0$  is a closed path based at  $\pi = (3, 2, 1)$ ,  $p_0$  admits length  $n_0$ , and associated with  $p_0$  is the matrix  $A^{(n_0)}$  with column summations  $a_1^{(n_0)}, a_2^{(n_0)}, a_3^{(n_0)}$  satisfying  $a_2^{(n_0)} + 1 = a_1^{(n_0)} + a_3^{(n_0)}$ . Then by similar computation as 1)(i) we get after going along  $p$ , the return times satisfy:

$$(2.3) \quad a_2^{(n_0+l+2)} = a_1^{(n_0+l+2)} + a_3^{(n_0+l+2)} - 1$$

2. Similarly as 1. above, the the same relation on the three return times is true for the path  $p = p_0 ba^l b$

By Case1 and Case2 We have proved Lemma 2.3.  $\square$

### 3 Whirly Three Interval Exchange Transformations

Let  $\pi = (321)$ , the symmetric 3-permutation, as the previous. According to W.A.Veech[16], there exists  $c_1, c_2, \dots, c_n \in \{a, b\}$ , such that:  $c_n \circ c_{n-1} \circ \dots \circ c_1(\pi) = \pi$ ;  
Let  $\pi^{(0)} = \pi$ ,  $\pi^{(1)} = c_1\pi^{(0)}$ ,  $\pi^{(2)} = c_2\pi^{(1)} \dots \pi^{(n)} = c_n\pi^{(n-1)} = \pi$ , let  $A^{(i)} = A(\pi^{(i-1)}, c_i)$ , ( $1 \leq i \leq n$ ), then  $B = A^{(1)}A^{(2)} \dots A^{(n)}$  is a positive  $m \times m$  matrix.

**Remark 3.1.** If  $\lambda \in \Lambda_m$ , then  $\mathcal{Z}^n(B\lambda, \pi) = (\lambda, \pi)$ , and the orbit of  $(B\lambda, \pi)$  under  $\mathcal{Z}$  passes the same sequence of permutations  $\{\pi^j, 0 \leq j \leq n\}$ .

Let  $\nu(A) = \max_{1 \leq i, j, k \leq m} \left\{ \frac{a_{ij}}{a_{ik}} \right\}$ , where  $A$  is a positive matrix, then:

i)

$$(3.1) \quad a_i \leq \nu(A)a_j, \quad 1 \leq i, j \leq m \quad (a_i \text{ is the } i\text{th column sum of } A)$$

ii)

$$(3.2) \quad \nu(BA) \leq \nu(A), \text{ for any nonnegative matrix } B.$$

$\nu(B)$  is a positive number greater than one.

Let  $r = \nu(B)$  and  $r' = \nu(B^t)$ .

Next we assume  $m = 3$ ,  $l$  be a given positive integer and we will set up an open set in  $\mathbb{L}_3 \times \{\pi\}$  and do some computation on the approximation by the Kakutani tower associated with the Rauzy induction.

Let  $\varepsilon_1, \varepsilon_2$  be two small positive numbers to be specified for our purpose later. Let  $Y^*(\varepsilon_1, \varepsilon_2) = \{\alpha \mid \alpha \in \mathbb{L}_m, (1 - \frac{\varepsilon_1}{2})|\alpha| > \alpha_2 > (1 - \varepsilon_1)|\alpha| \text{ and } (1 + \varepsilon_2)\alpha_3 > \alpha_1 > \alpha_3\}$ , thus  $Y^*(\varepsilon_1, \varepsilon_2)$  is an open subset of  $\mathbb{L}_m$ . Let  $Y(\varepsilon_1, \varepsilon_2) \times \{\pi\}$ , and  $W(\varepsilon_1, \varepsilon_2) = (B^2Y^*(\varepsilon_1, \varepsilon_2)) \times \{\pi\}$ .  $W(\varepsilon_1, \varepsilon_2)$  is an open subset of  $\mathbb{L}_m \times \{\pi\}$ .

Suppose  $(\lambda, \pi) \in \Delta_2 \times \{\pi\}$ , and there exists  $k \in \mathbb{N}$  such that  $\mathcal{Z}^k(\xi, \pi) \in W(\varepsilon_1, \varepsilon_2)$ . We know  $\xi = B^2\alpha$  for some  $\alpha \in \Lambda_3$

Suppose  $\lambda = A^{(k)}\xi$ , where  $A^{(k)}$  is the visitation matrix associated with  $\mathcal{Z}^k(\lambda, \pi)$ . Then  $\mathcal{Z}^{k+2n}(\lambda, \pi) = \mathcal{Z}^{2n}(\xi, \pi) = (\alpha, \pi)$ , and  $\lambda = A^{(k)}B^2\alpha$ . Let  $A = A^{(k)}B^2$ . Since  $A^{(k)}$  is a non-negative matrix, by 3.2 we have  $\nu(A) \leq \nu(B) = r$ . Therefore the following may give us a clear view of the stack structure associated with the Veech-induction map  $(\mathcal{T}_2)$ :

- 1) Translating the subinterval  $I_2^\alpha$  (i.e. the second subinterval of  $I^\alpha$ ) to the left by  $(\alpha_1 - \alpha_3)$ , we get  $T^{a_2}(I_2^\alpha)$ . Therefore

$$I_2^\alpha \cap T^{a_2}(I_2^\alpha) = [\alpha_1, \alpha_1 + \alpha_2 - l(\alpha_1 - \alpha_3))$$

Thus, since  $l$  is a fixed positive integer, and  $\varepsilon_1, \varepsilon_2$  are small enough, we have

$$(3.3) \quad \begin{aligned} \mu(I_2^\alpha \cap T^{la_2}(I_2^\alpha)) &= \alpha_2 - l(\alpha_1 - \alpha_3) > \alpha_2 - l\varepsilon_2\alpha_3 \\ &> \alpha_2 - l\varepsilon_1\varepsilon_2|\alpha| > \alpha_2 - l\frac{\varepsilon_1\varepsilon_2}{1-\varepsilon_1}\alpha_2 \\ &= (1 - l\frac{\varepsilon_1\varepsilon_2}{1-\varepsilon_1})\alpha_2 \end{aligned}$$

- 2) The remainder of the column with base  $I_2^\alpha$  and height  $a_2$  has measure:



$$\begin{aligned}
(3.4) \quad & |\lambda| - \mu(\cup_{i=0}^{a_2} T^i(I_2^\alpha)) = \\
& a_1\alpha_1 + a_3\alpha_3 \\
& < r\alpha_2(\alpha_1 + \alpha_3) \\
& < ra_2\varepsilon_1 |\alpha| < ra_2 \frac{\varepsilon_1}{1-\varepsilon_1} \alpha_2 < r \frac{\varepsilon_1}{1-\varepsilon_1} |\lambda|
\end{aligned}$$

Which with 1) tell us that ,  $\{la_2\}$  is a sequence such that  $\lim T^{la_2} = Id$ .

- 3)** (the 'whirly part')  $T^{a_3}$  sends  $[\alpha_1 + \alpha_2 + (\alpha_1 - \alpha_3), |\alpha|]$  to  $[0, 2\alpha_3 - \alpha_1]$  which is continuous under  $T^{a_1}$ . That is to say  $[\alpha_1 + \alpha_2 + (\alpha_1 - \alpha_3), |\alpha|]$  is continuous under  $T^{a_1+a_3} = T^{a_2-1}$ .

Similarly by induction:

Let

$$(3.5) \quad I_\omega^\alpha = [\alpha_1 + \alpha_2 + l(\alpha_1 - \alpha_3), |\alpha|]$$

Then  $T^i$  are all continuous (linear) on  $I_\omega^\alpha$  for  $i = 1, 2, \dots, l(a_1 + a_3)$ . And  $T^{l(a_1+a_3)}(I_\omega^\alpha) \subset I_3^\alpha \subset I^\alpha$ . Therefore

$$\begin{aligned}
T^{la_2}(I_\omega^\alpha) &= T^{l(a_1+a_3-1)}(I_\omega^\alpha) \\
&= T^{-l}(T^{l(a_1+a_3)}(I_\omega^\alpha)) \subset T^{-l}(I^\alpha)
\end{aligned}$$

This implies

$$(3.6) \quad T^{la_2}(I_\omega^\alpha) \subset (I^{la_2}(I^\alpha)) \cap T^{-l}(I^\alpha)$$

then

$$\begin{aligned}
(3.7) \quad & \mu(T^{la_2}(I^\alpha) \cap T^{-l}(I^\alpha)) \geq \mu(T^{la_2}(I_\omega^\alpha)) \\
& = \alpha_3 - l(\alpha_1 - \alpha_3) > \alpha_3 - l\varepsilon_2\alpha_3 \\
& = (1 - l\varepsilon_2)\alpha_3
\end{aligned}$$

Note: **1)** and **2)** show that  $T^{la_2}$  is close to the identity map; 3.7 shows that we are on the right way to whirly (Definition 1.13) .

By **1)** **2)** and **3)** , choose a constant  $C_{\varepsilon,l}$  associated with  $\varepsilon, l$  and small enough, we have:

**Lemma 3.2.** *All the notations as above, for any  $0 < \varepsilon < \frac{1}{10}$ ,  $l \in \mathbb{N}$ , there exists  $C_{\varepsilon,l}$  small enough such that for any  $(\lambda, \pi) \in \Lambda_3 \times \{\pi\}$  satisfying  $\mathcal{Z}^k(\lambda, \pi) = (\eta, \pi) \in W(C_{\varepsilon,l}, C_{\varepsilon,l})$  we have: suppose  $\mathcal{Z}^{k+2n}(\lambda, \pi) = (\alpha, \pi)$ , then*

$$\begin{aligned}
P1) & \dots \mu(I^\alpha \cap T^{la_2}(I^\alpha)) > (1 - \varepsilon) |\alpha| \\
P2) & \dots |\lambda| - \mu(\cup_{i=0}^{a_2-1} T^i(I_2^\alpha)) < \varepsilon |\lambda| \\
P3) & \dots \mu(T^{la_2}(I^\alpha) \cap T^{-l}(I^\alpha)) > \frac{\varepsilon}{3} |\alpha|
\end{aligned}$$

Now let  $N_{\varepsilon,l}^{(\lambda)}$ , a subset of  $\mathbb{N}$ , be defined by

$$(3.8) \quad N_{\varepsilon,l}^{(\lambda)} = \{n_t | n_1 < n_2 < \dots < n_i < \dots, \mathcal{Z}^{n_t-n}(\lambda, \pi) \in W(C_{\varepsilon,l}, C_{\varepsilon,l})\}$$

By Veech's Ergodic Theorem W.A.Veech[15] on  $\mathcal{T}_2$ , we know that for Lebesgue a.e.  $\lambda \in \mathbb{L}$ ,  $T(\lambda, \pi)$  is unique ergodic (thus ergodic with respect to Lebesgue measure), and  $N_{\varepsilon,l}^{(\lambda)}$  is a set with infinitely many elements, for any  $0 < \varepsilon < \frac{1}{10}$ ,  $l \in \mathbb{N}$ . Lets continue to study such  $T_{(\lambda,\pi)}$ . As usual sometimes we use  $T$  to denote  $T_{\lambda,\pi}$ .

**Density Point Argument and the Proof of Theorem 1.15** We know that  $A = A^{(k)}B^2$ ,  $1 \leq \nu(A) \leq \nu(B) = r$ . We need  $B^2$  here instead of  $B$ , in order to get a  $T$ -stack with the base  $I^\alpha$ , which is a relatively large portion of  $I^\lambda$ .

**Lemma 3.3.** *All notations as above, there exists an positive integer  $a_*$  such that  $T^i$  ( $1 \leq i \leq a_*$ ) are continuous (linear) on  $I^\alpha$ ,  $T^i(I^\alpha) \cap T^j(I^\alpha) = \Phi$ , ( $i \neq j, 0 \leq i, j < a_*$ ), and*

$$a_* |\alpha| > \frac{1}{b_M(1 + 2r \cdot r')} |\lambda|$$

where  $b_M = \max\{b_{11}, b_{12}, b_{13}\}$ .

*Proof.* We know that  $A = A^{(k)}B^2$ .

Suppose  $\mathcal{Z}^{k+n}(\lambda, \pi) = (\eta, \pi) = (B\alpha, \pi)$ , where as usual  $\eta = (\eta_1, \eta_2, \eta_3)$ ,  $I_1^\eta = [0, \eta_1]$  then We have  $\eta_1 = b_{11}\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3$ , and

$$(3.9) \quad I^\alpha \subset I_1^\eta$$

Meanwhile  $\eta_1 < b_M(\alpha_1 + \alpha_2 + \alpha_3) < b_M |\alpha|$ , that is

$$(3.10) \quad |\alpha| > \frac{1}{b_M} \eta_1$$

At the same time since

$$\begin{aligned} \eta_1 &= b_{11}\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3 \\ \eta_2 &= b_{21}\alpha_1 + b_{22}\alpha_2 + b_{23}\alpha_3 \\ \eta_3 &= b_{31}\alpha_1 + b_{32}\alpha_2 + b_{33}\alpha_3 \end{aligned}$$

thus:

$$(3.11) \quad \eta_2, \eta_3 < r' \eta_1$$

Reminding that  $\lambda = A^{(k)}B$ , and  $\lambda = a_1^{(k+n)}\eta_1 + a_2^{(k+n)}\eta_2 + a_3^{(k+n)}\eta_3$ , by 3.1 and 3.2 we have

$$a_2^{k+n}, a_3^{k+n} < r a_1^{k+n}$$

by 3.11, we have

$$(3.12) \quad a_1^{(k+n)}\eta_1 > \frac{1}{1 + 2rr'} \lambda$$

3.10 and 3.11 implies  $a_1^{k+n}\alpha > \frac{1}{b_M(1+rr')} \lambda$ , combining with 3, the Lemma is proved.  $\square$

### Proof of Theorem 1.15

*Proof.* Define  $G = \bigcap_{N=1}^{\infty} \bigcup_{\substack{t \leq N \\ n_t \in N_{\varepsilon, l}^{(\lambda)}}} G_t$ , where  $G_t = \bigcup_{i=0}^{a_*^{(n_t)}-1} T^i(I^{\alpha^{n_t}})$

Then it is obvious that  $\mu(G) \geq \frac{1}{b_M} \frac{1}{1 + 2rr'} |\lambda|$ .

Suppose  $E$  is an arbitrary measurable set,  $E \subset [0, |\lambda|]$ ,  $\mu(E) > 0$ . Then there exists  $q \in \mathbb{N}$  such that  $\mu(T^{-q}(E) \cap G) > 0$ . Therefore there exists a density point  $x \in T_{(\lambda, \pi)}^{-q}(E) \cap J_k$ , where  $(J_k = T^{i_k}(I^{\alpha^{S_k}}))$ , where  $S_k = n_{t_k}$  ( $0 \leq i_k < a_*^{S_k}$ ), and

$$(3.13) \quad \lim_{k \rightarrow \infty} \frac{\mu(T^{-q}(E) \cap J_k)}{\mu(J_k)} = 1$$

By Lemma 3.2 , we know that since  $S_k = n_{t_k} \in N_{\varepsilon,l}^{(\lambda)}$

$$(3.14) \quad \begin{aligned} & \mu((T^{la_2^{S_k}} J_k) \cap T^{-l}(J_k)) \\ & \dots\dots\dots = \mu(T^{i_k}(T^{la_2^{(S_k)}}(I^{\alpha^{(S_k)}}) \cap T^{-l}(I^{\alpha^{(S_k)}}))) \\ & > \frac{\varepsilon}{3} |\alpha^{S_k}| \end{aligned}$$

3.13 implies there exists  $k_0$  such that

$$\frac{\mu(T^{-q}(E) \cap J_{k_0})}{\mu(J_{k_0})} > (1 - \frac{\varepsilon}{10})$$

Therefore by 3.14 we have

$$\mu(T^{la_2^{(S_k)}}(T^{-q}(E)) \cap (T^{-q}(E))) > 0$$

Thus  $\mu(T^{la_2^{(S_k)}}) > 0$ . that is  $E \cap T^{-l}(E) > 0$ . Since  $S_k = n_{t_k} \in N_{\varepsilon,l}^{(\lambda)}$ , therefore we have proved Theorem 1.14: □

**Corollary 3.4.** *For Lebesgue almost all  $\lambda \in \Lambda_3$ ,  $(\mathbb{X}, \mathcal{B}, T_{(\lambda\pi)})$  admits no spatial model.*

#### Comments and Acknowledgements

The result about 3-interval exchange transformations is a part of the author Y. Wu's 2006 Ph.D. Thesis at Rice University , Department of Mathematics. The author Y. Wu thanks Rice University to collect his Doctor of Philosophy Thesis in the Rice University Electronic Theses and Dissertations online. He would like to thank his Ph.D. advisor W. A. Veech for directing his Ph.D. Thesis.

## References

- [1] A. Avila, G. Forni, *Weak mixing for interval exchange transformations and translation flows*. Ann of Math, Vol 165 (2007), Issue 2, 637-664
- [2] M. Boshernitzan *A condition for minimal interval exchange maps to be uniquely ergodic*. Duke Math. J. 52 (1985), no. 3, 723–752.
- [3] R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*. Proc. Amer. Math. Soc. 22 1969 559–562.
- [4] J. Chaika, *Every transformation is disjoint from almost every IET*, arXiv:0905.2370(2009)
- [5] E. Glasner, B. Tsirelson, B. Weiss, *The automorphism group of the Gaussian measure cannot act pointwise*. Isr. J. Math. 148 (2005), 305-329
- [6] E. Glasner, *Ergodic theory via joinings*. Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.
- [7] E.Glasner, B.Tsirelson, B.Weiss, *The automorphism group of the Gaussian measure cannot act pointwisely* Israel J. of Math.. Dec 2005 Vol 148 Iss. 1 pp305-329.
- [8] E.Glasner, B.Weiss, *G-continuous functions and whirly actions*. Preprint: ArXiv math.DS/0311450.
- [9] P.R. Halmos, *Measure Theory*. Graduate Texts in Mathematics 18, Springer-Verlag.
- [10] P.R.Halmos, *Introduction to Ergodic Theory*. New York Press.
- [11] M. Keane, *Interval Exchange transformations*. Math Z. 141(1973), 25-31.

- [12] J.L. King, *The commutant is the weak closure of the powers, for rank-1 transformations.* Ergodic Theory Dynam. Systems 6 (1986) no. 3, 363–384.
- [13] H.Masur, *Interval Exchange Transformation and measrued foliation* Ann. of Math., 115 169-200.
- [14] G. Rauzy, *Echanges d'intervalles et transformations induites.* Acta Arith 34(1979) 315-328.
- [15] W.A. Veech, *Gauss Measures for Transformations on the Space of Interval Exchange Map.* Ann of Math, Vol 115(1982), 201-242.
- [16] W.A. Veech *The Metric Theory of Interval Exchange Transformation I : Generic Spectral Properties.* American Journal of Mathematics, 107(6):1331-1359,1984.
- [17] W.A. Veech *Interval exchange transformations.* J.D. Analyse Math. 33(1978) 222-278.
- [18] Y. Wu, *Applications of Rauzy Induction on the generic ergodic theory of interval exchange transformations*, Doctor of Philosophy Thesis, Rice University Electronic Theses and Dissertations(2006)